Econometrics Spring School 2016
Econometric Modelling

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Lecture 9: Non-linear Model Selection

Motivation:

- Non-linearity inherent in economics (and many other fields)
- Linear representation is a simplifying assumption
- If incorrect, model will be mis-specified

Objective:

- Test for evidence of non-linearity
- Commence with general non-linear approximation.
  Investigate large class of functions: polynomials one possibility:
    - needs to be identified and congruent;
    - approximate wide range of non-linear models;
    - maintains linearity in the parameters;
    - easy to orthogonalize.
- But non-generalized polynomials not invariant to transformations

Establish operating characteristics

Non-linear model selection:

- Testing for non-linearity;
  - Caveat: ...like going to the zoo to look at non-elephants[1]
- Mimicking a near-orthogonal representation;
- Avoiding extreme observations leading to non-normality – impulse-indicator saturation and step-indicator saturation;
- Preventing excess retention of irrelevant variables – choice of significance level.

A successful algorithm requires the synthesis of all developments to be implemented

[1] With thanks to Anders Rahbek.

Assume LDGP is:

$$y_t = f\left(z_{1,t}, \ldots, z_{k,t}; \theta\right) + \epsilon_t \quad \text{where} \quad \epsilon_t \sim \mathsf{IN}\left[0, \sigma_\epsilon^2\right] \tag{1}$$

for $t = 1, \ldots, T$, with $\theta \in \boldsymbol{\Theta} \subseteq \mathbb{R}^k$.

Problems include:

- specification of functional form, $f\left(\cdot\right)$;
- identification of θ;
- selection of relevant variables, $\mathbf{z}_t' = \left(z_{1,t}, \ldots z_{k,t}\right)$ from available candidates $\left(z_{1,t}, \ldots z_{K,t}\right)$ where $K \geq k$.

Specify a GUM that nests the LDGP to ensure the initial formulation is congruent:

$$y_t = \sum_{j=1}^{K} \beta_j \sum_{p=1}^{P} h_p \left(z_{1,t}, \ldots z_{K,t} \right) + \nu_t \quad \text{where} \quad \nu_t \sim \mathsf{IN}\left[0, \sigma_\nu^2\right] \qquad (2)$$

k relevant and $K - k$ irrelevant variables.
P approximation bases.
Test for non-linearity to see if it is viable to reduce to:

$$y_t = \sum_{j=1}^{K} \beta_j z_{j,t} + \nu_t. \qquad (3)$$

If do not reject, proceed with linear GUM.
If reject, non-linearity is established.

2 cases:

1. **GUM nests LDGP**
   Approximation is exact, $f(\cdot) \subseteq h_p(\cdot)$
   Congruency is meaningful and testable, even with more variables than observations
   Consistent model selection: $\alpha \to 0$ as $T \to \infty$ so irrelevant variables eventually retained with prob 0.
   Functional form concerns whether more parsimonious representation found.
2. **GUM doesn't nest LDGP but is approximation**
   Not clear what consistency means – test of approx.
   Cannot prove consistency as non-nested LDGP.
   Consistency feasible with encompassing test stage.
   Congruency still operational.

Commence with general approximation $h_p(\mathbf{z}_i)$:

$$y_t = \beta'\mathbf{z}_t + \gamma'\mathbf{g}(\mathbf{w}_t) + \sum_{t=1}^{T} \delta_t \mathbf{1}_{\{t=t_i\}} + v_t \quad t = 1, \ldots, T \tag{4}$$

where $v_t \sim \mathsf{NID}\left(0, \sigma_v^2\right)$.
$\mathbf{z}_t = (z_{1,t}, \ldots, z_{K,t})'$ is $(K \times 1)$ vector of potentially relevant variables.
$\mathbf{g}(\mathbf{w}_t)$ is $(M \times 1)$ vector of non-linear transformations on standardized potentially relevant variables or factors, $\mathbf{w}_t$:

$$w_{j,t} = \frac{z_{j,t} - \overline{z}_j}{\sigma_{z_j}} \qquad j = 1, \ldots, K, \quad \text{or} \quad \mathbf{w}_t = \mathbf{\Lambda}^{1/2}\mathbf{H}'(\mathbf{z}_t - \overline{\mathbf{z}}) \tag{5}$$

$\sum_{t=1}^{T} \mathbf{1}_{\{t=t_i\}}$ is a set of saturating indicators.

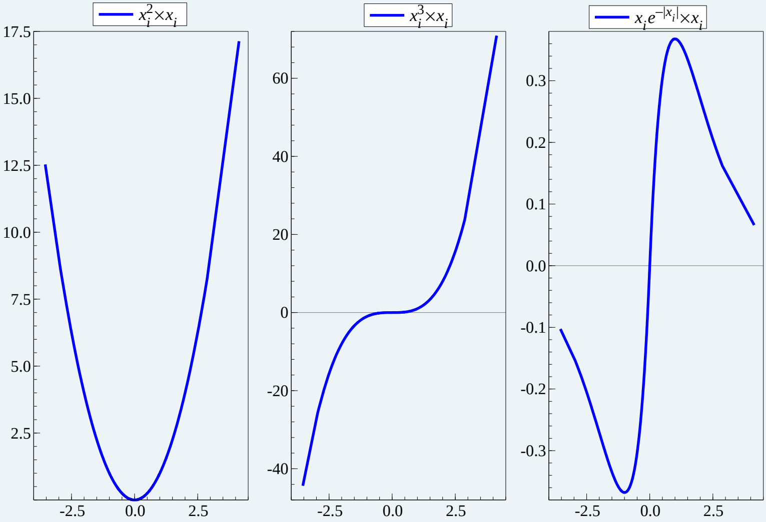
Key question: What $\mathbf{g}(\mathbf{w}_t)$ specification?

Many possible approximations:

- Polynomials, Hermite, Chebyshev, ...
- Fourier series approximations
- Asymptotic series
- Logistic functions

Range of different bases – different shapes of functions.
Ability to parsimoniously approximate depends on LDGP.
Suggests many bases but only first few terms of each.
Preferable to one base with longer approximation.
Issue: approximations not orthogonal – could 'cancel', hence selection.

$$\begin{aligned}\gamma'\mathbf{g}(\mathbf{w}_t) &= \sum_{j=1}^{K}\sum_{l=j}^{K}\beta_{jl}w_{j,t}w_{l,t} && \text{[2}^{\text{nd}}\text{ order polynomials]} \\ &+ \sum_{j=1}^{K}\sum_{l=j}^{K}\sum_{q=l}^{K}\beta_{jlq}w_{j,t}w_{l,t}w_{q,t} && \text{[3}^{\text{rd}}\text{ order polynomials]} \\ &+ \sum_{j=1}^{K}\alpha_{jj}\left\{w_{j,t}e^{-|w_{j,t}|}\right\} && \text{[exponentials]} \qquad (6)\end{aligned}$$


$x_i^2 \times x_i$
17.5
15.0
12.5
10.0
7.5
5.0
2.5
-2.5
0.0
2.5
$x_i^3 \times x_i$
60
40
20
0
-20
-40
-2.5
0.0
2.5
$x_i e^{-|x_i|} \times x_i$
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
-2.5
0.0
2.5

Selection undertaken on (11):

$$\widehat{y}_t = \sum_{j=1}^{k^*} \widehat{\beta}_j z_{j,t} + \sum_{j=1}^{m^*} \widehat{\gamma}_j g_j\left(\mathbf{w}_t\right) + \sum_{i=1}^{q} \widehat{\delta}_i \mathbf{1}_{\{t=t_i\}} \tag{7}$$

k^* = no. linear regressors retained;
m^* = no. non-linear transformations retained;
q = indicators retained.
Final stage – test approximation (7) against preferred functional form, $\psi\left(\mathbf{z}_t\right)$ (e.g. LSTAR, theory-motivated etc).
Encompassing test: $H_0 : \gamma_j = 0, \forall j$:

$$y_t = \sum_{j=1}^{k^*} \widehat{\beta}_j z_{j,t} + \sum_{j=1}^{m^*} \gamma_j g\left(\mathbf{w}_t\right) + \sum_{i=1}^{q} \delta_i \mathbf{1}_{\{t=t_i\}} + \lambda' \psi\left(\mathbf{z}_t\right) + \eta_t. \tag{8}$$

We shall examine each aspect in turn before undertaking selection jointly.
In practice, all aspects should be implemented jointly.

1. Test for non-linearity
2. Outlier detection
3. Non-linear functions
4. Joint selection using a parsimonious non-linear function
5. Use of SIS to obtain an encompassing model

Test for non-linearity in general linear model by low-dimensional portmanteau test in Castle and Hendry (2010) uses cubics of principal components $\boldsymbol{w}_t$ of the $\boldsymbol{z}_t$.

Let $\boldsymbol{z}_t \sim \mathsf{D}_n[\boldsymbol{\mu}, \boldsymbol{\Omega}]$, where $\boldsymbol{\Omega} = \boldsymbol{H}\boldsymbol{\Lambda}\boldsymbol{H}'$ with $\boldsymbol{H}'\boldsymbol{H} = \boldsymbol{I}_n$.

Then $\boldsymbol{w}_t^* = \boldsymbol{H}'\boldsymbol{z}_t \Rightarrow \boldsymbol{w}_t^* \sim \mathsf{D}_n[\boldsymbol{H}'\boldsymbol{\mu}, \boldsymbol{\Lambda}]$. Empirically:

$\widehat{\boldsymbol{\Omega}} = T^{-1}\sum_{t=1}^{T}(\boldsymbol{z}_t - \overline{\boldsymbol{z}})(\boldsymbol{z}_t - \overline{\boldsymbol{z}})' = \widehat{\boldsymbol{H}}\widehat{\boldsymbol{\Lambda}}\widehat{\boldsymbol{H}}'$ so that $\boldsymbol{w}_t = \widehat{\boldsymbol{\Lambda}}^{-1/2}\widehat{\boldsymbol{H}}'(\boldsymbol{z}_t - \overline{\boldsymbol{z}})$ leading to $\boldsymbol{w}_t \underset{\text{app}}{\sim} \mathsf{D}_n[\boldsymbol{0}, \boldsymbol{I}]$.

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If test rejects, create $\boldsymbol{g}(\boldsymbol{w}_t)$, otherwise $\boldsymbol{g}(\boldsymbol{z}_t) = \boldsymbol{z}_t$: presently, implemented general cubics with exponential functions.
$u_{1,i,t} = w_{i,t}^2;\ u_{2,i,t} = w_{i,t}^3;\ u_{3,i,t} = w_{i,t}e^{-|w_{i,t}|}$.

When $\boldsymbol{\Omega}$ is non-diagonal, each $w_{i,t}$ is a linear combination of every $z_{i,t}$, so $w_{i,t}^2$ involves squares and cross-products of every $z_{i,t}$ etc.

Advantages of test are:

- Low dimensionality–may entail more non-linear functions than observations;
- No collinearity between elements of $\boldsymbol{w}_t$;
- Includes most important sources of departure from linearity, e.g. asymmetry.

Number of potential regressors for cubic polynomials is:

$$M_K = K\left(K+1\right)\left(K+5\right)/6.$$

Explosion in number of terms as $K = r \times (s+1)$ increases:

K	1	2	3	4	5	10	15	20	30	40
M_K	3	9	19	30	55	285	679	1539	5455	12300

Quickly reach huge M_K: **but only** $3K$ **if use** $w^k_{i,t-j}$.
Later address perfect collinearity between $\boldsymbol{z}_t$ and $\boldsymbol{w}_t$.

Non-Linear Model Selection in Practice

- Load data "**nonlin_nobreak_example.in7**"
- Batch file: "**nonlinear_data_example.fl**"
- Artificial data with non-linear DGP:

$$\text{DGP: } y_t = \beta_1 x_{1,t} + \beta_2 x_{1,t}^2 + \beta_3 x_{1,t}^3 + \beta_4 x_{2,t}^2 + \epsilon_t \tag{9}$$

where $\mathsf{E}[\psi_i] = t_{\beta_i} = 3$ for $i = 1, \ldots, 4$, and $\epsilon_t \sim N(0, 1)$.

- **Estimate the mis-specified model**:

$$y_t = \beta_0 + \sum_{i=1}^{10} \beta_i x_{i,t} \tag{10}$$

The non-linearity index test applied to a linear model of y_t, where the regressors include an intercept, $x_{1,t}$ to $x_{10,t}$:

The test is significant at $\mathsf{p} = 0.008$ with $\mathsf{F}(30, 59) = 2.10$.

Specify GUM $\rightarrow$ Estimate model.
Test $\rightarrow$ Index test for non-linearity.
Quadratic, cubic and exponential principal components included.

Caveat: model also fails other diagnostics – must jointly test for outliers using IIS/SIS (Castle, Doornik, Hendry, and Pretis (2015)).

- K=10 variables → 285 regressors if all non-linear combinations (highly correlated)
- Principal components: low dimensionality

Create principal components of set of explanatory variables:

$$(x_{1,t}, \ldots, x_{10,t})$$

Other models → Descriptive statistics using PcGive.
Select regressors and choose Principal component analysis. Make sure you select the Save components in database option.

Compute squares and cubics of principal components using calculator/algebra:

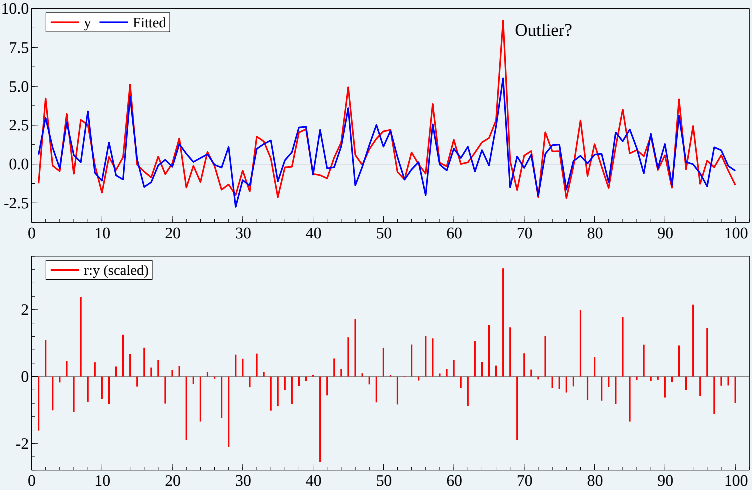
- `PC1sq = PC1^2;`
- `PC1cub = PC1^3;`
- `PC1exp = PC1*exp(-abs(PC1));`
- ...

Specify GUM (up to PC3) and select at tight significance level, $p_\alpha = 0.01$

Model Selection using non-linear transformations of principal components:

	Coefficient	Std.Error	t-value
x1	1.50656	0.1235	12.2
PC1_sq	0.160795	0.04795	3.35
PC3_sq	0.147337	0.05937	2.48

```
AR 1-2 test:      F(2,95)   =  0.77799 [0.4622]
ARCH 1-1 test:    F(1,98)   = 0.027482 [0.8687]
Normality test:   Chi^2(2)  =   3.6125 [0.1643]
Hetero test:      F(6,93)   =   2.8462 [0.0137]*
Hetero-X test:    F(9,90)   =   5.8234 [0.0000]**
RESET23 test:     F(2,95)   =   13.508 [0.0000]**
```



Must tackle non-linearities and shifts jointly!

Extend GUM to include Step-Indicators, select at $p_{\alpha=0.01}$:

$$y_t = \beta' \mathbf{z}_t + \gamma' \mathbf{g}(\mathbf{w}_t) + \sum_{t=1}^{T} \delta_t \mathbf{1}_{\{t \leq t_i\}} + v_t \quad t = 1, \ldots, T \qquad (11)$$

Results:

	Coefficient	Std.Error	t-value	t-prob
x1	1.34988	0.1223	11.0	0.0000
PC1_sq	0.142090	0.04767	2.98	0.0036
S1:66	-4.94129	1.138	-4.34	0.0000
S1:67	5.15142	1.145	4.50	0.0000

```
AR 1-2 test:       F(2,94)    =   1.3612 [0.2614]
ARCH 1-1 test:     F(1,98)    =  0.29562 [0.5879]
Normality test:    Chi^2(2)   =   1.0758 [0.5840]
Hetero test:       F(5,93)    =  0.54736 [0.7399]
Hetero-X test:     F(6,92)    =  0.58405 [0.7422]
RESET23 test:      F(2,94)    =   4.0001 [0.0215]*
```

Tackling non-linearities and shifts jointly.

- **So far**: non-linear DGP approximated by non-linear principal component + step-shifts.
- **Now**: test against preferred functional form
- Final stage – test approximation against preferred functional form, $\psi(\mathbf{z}_t)$ (e.g. LSTAR, theory-motivated etc).

Encompassing test: $H_0 : \gamma_j = 0, \forall j$:

$$y_t = \sum_{j=1}^{k^*} \widehat{\beta}_j x_{j,t} + \sum_{j=1}^{m^*} \gamma_j g(\mathbf{w}_t) + \sum_{i=1}^{q} \delta_i \mathbf{1}_{\{t \leq t_i\}} + \lambda' \psi(\mathbf{x}_t) + \eta_t.$$

Use 'theory motivated' specification:

$$\lambda' \psi(\mathbf{x}_t) = \lambda_1 x_{1,t}^2 + \lambda_2 x_{1,t}^3 + \lambda_3 x_{2,t}^2$$

Encompassing Model:

$$y_t = \beta_1 x_{1,t} + \gamma_1 w_{1,t}^2 + \delta_1 S_{t=66} + \delta_2 S_{t=67} + \lambda_1 x_{1,t}^2 + \lambda_2 x_{1,t}^3 + \lambda_3 x_{2,t}^2$$

```
Test for excluding:
[0] = PC1_sq
[1] = S1:66
[2] = S1:67
Subset F(3,93)   =  0.49576 [0.6861]
```

Detected shifts drop out when non-linear functional form specified.

Effect of using SIS when selecting from GUM nesting a non-linear DGP

Retention at 1% (x's selected over)		
	without SIS	with SIS
x, $\psi = 3$	0.68	0.63
x^2, $\psi = 3$	0.68	0.56
x^3, $\psi = 3$	0.67	0.62
SIS gauge	-	0.03

Little effect when DGP is truly non-linear.

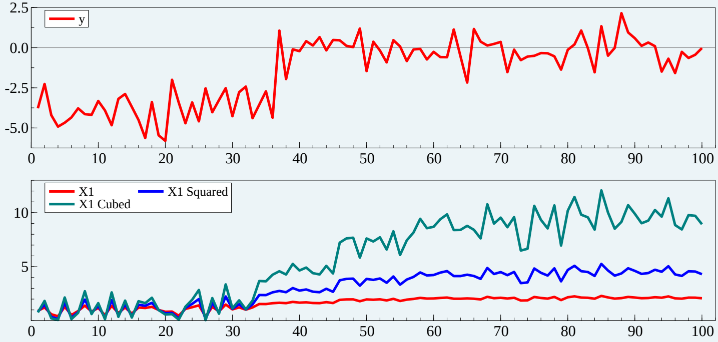
Non-Linear Example 2: Breaks

So far:

- Automatic non-linear extensions
- SIS steps not retained if DGP truly non-linear

Now:

- Spurious non-linearity due to shifts
- Load dataset: "nonlin_break_example.in7"



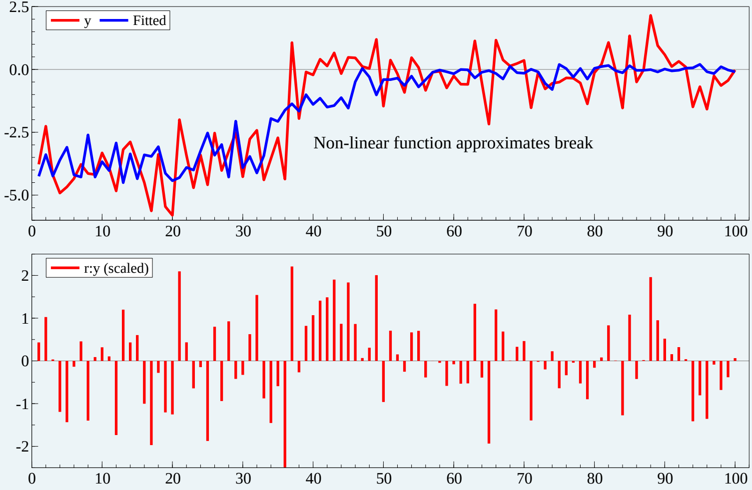
Select model at $p_\alpha = 0.01$ starting from GUM:

$$y_t = \beta_1 x_{1,t} + \beta_2 x_{1,t}^2 + \beta_3 x_{1,t}^3 + \sum_{i=2}^{5} \lambda_i x_{i,t}$$

Selection results:

	Coefficient	Std.Error	t-value	t-prob
x1	-13.4025	1.139	-11.8	0.0000
x3	0.0879400	0.1070	0.822	0.4131
x1_sq	11.9787	1.437	8.33	0.0000
x1_cub	-2.66853	0.4369	-6.11	0.0000

```
AR 1-2 test:        F(2,94)    =   0.80101 [0.4519]
ARCH 1-1 test:      F(1,98)    =    1.0423 [0.3098]
Normality test:     Chi^2(2)   =0.0010385 [0.9995]
Hetero test:        F(7,92)    =    2.6783 [0.0144]*
Hetero-X test:      F(11,88)   =    1.6932 [0.0880]
RESET23 test:       F(2,94)    =    10.153 [0.0001]**
```



Spurious non-linearity?

Start with GUM including step-indicators:

$$y_t = \beta_1 x_{1,t} + \beta_2 x_{1,t}^2 + \beta_3 x_{1,t}^3 + \sum_{i=2}^{5} \lambda_i x_{i,t} + \sum_{i=1}^{q} \delta_i \mathbf{1}_{\{t \leq t_i\}}$$

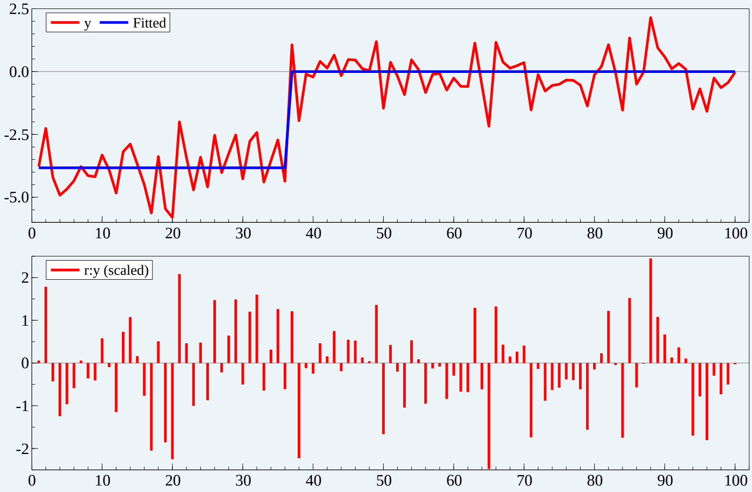
Yields:

	Coefficient	Std.Error	t-value	t-prob
S1:36	-3.82795	0.1463	-26.2	0.0000

```
AR 1-2 test:        F(2,97)    = 0.060408 [0.9414]
ARCH 1-1 test:      F(1,98)    =  0.30641 [0.5812]
Normality test:     Chi^2(2)   =  0.18369 [0.9122]
RESET23 test:       F(1,98)    =6.4913e-028 [1.0000]
```

Which nearly coincides with the DGP:

$$y_t = \mu + \lambda 1_{t \geq 35} + \epsilon_t \qquad (12)$$



Step-shift identified, spurious non-linearity removed.

SIS when non-linear transformations spuriously approximate a break of magnitude λ_1 in a linear DGP:

	Retention at 1% (x's selected over)			
	$\lambda_1 = 2\sigma_\epsilon$		$\lambda_1 = 4\sigma_\epsilon$	
	without SIS	with SIS	without SIS	with SIS
x, $\psi = 0$	0.41	0.02	0.83	0.02
x^2, $\psi = 0$	0.66	0.02	0.96	0.02
x^3, $\psi = 0$	0.3	0.02	0.83	0.01
T_1 step	-	0.62	-	0.94
SIS gauge	-	0.02	-	0.02

General Approach – Model Discovery Extensions:

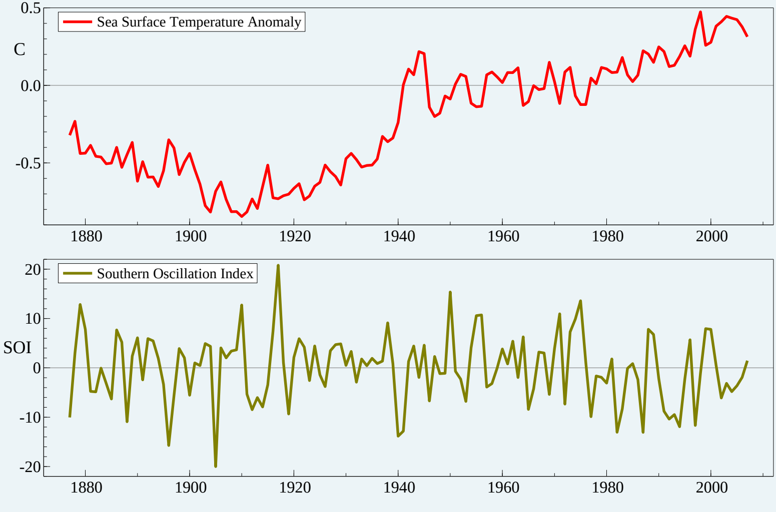
- Automatic non-linear extensions
- Jointly with indicator saturation

SIS as insurance mechanism:

- Non-linearities present: little effect of using SIS, steps not retained.
- Unknown shifts: identified through SIS rather than attributed to non-linearities

Global SST – Climate/Weather Indicator

Shape of trend? Breaks in series? El Niño Southern Oscillation?



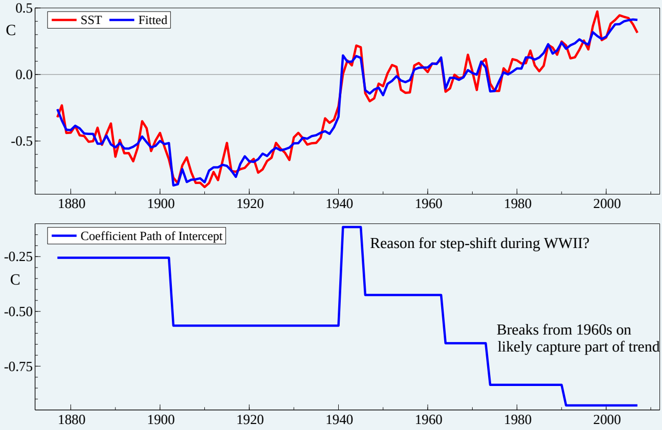
Model:

$$\mathbf{y_t} = \mathbf{f}(\mathbf{z_t}) + \beta \mathbf{x_t} + \delta \mathbf{D} + \epsilon_\mathbf{t} \quad (13)$$

- **T**=131 (1877-2007)
- **y**= Global Mean SST Anomalies (C) relative to 1950-79
- **z**=trend, **x**= Southern Oscillation Index (SOI) (atm. pressure at SL)
- **N** = 131 + 6 = 137 variables

Specification:

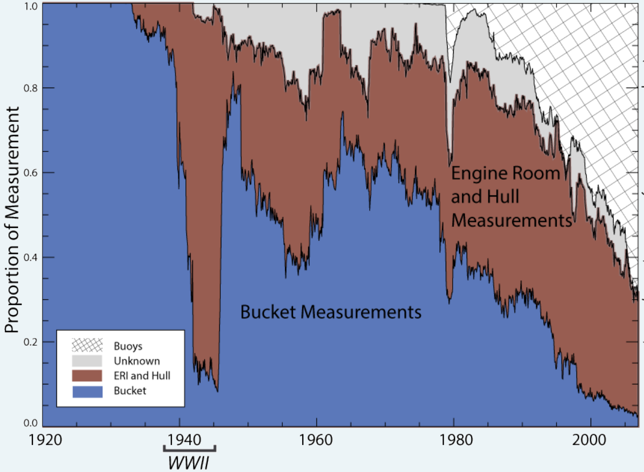
- SIS, $p_\alpha = 0.01$ (1%)
- Non-linear trend: B-spline basis (5 degree polynomial)



Multiple breaks, note:

- **1940**, $\hat{\delta}_1 = \mathbf{0.45^{***}C}$ (se=0.04)
- **1945**, $\hat{\delta}_2 = \mathbf{-0.31^{***}C}$ (se=0.04)

- WWII: **1941/1942 Measurements**: buckets to engine intake
  - Danger of measurements (light)
  - Americans joined 1941/1942
- Post-WWII: Partly changed back



(Adapted from Kennedy et al. 2012, *J. Geoph. Res.*)

- WWII: **1941/1942** **Measurements**: buckets to engine intake
  - Danger of measurements (light)
  - Americans joined 1941/1942
- Post-WWII: Partly changed back
- Buckets: **cold bias** ($\approx 0.3C$) (Matthews, 2012)
  - SIS: $\hat{\delta}_1 = 0.45$ (± 0.04)

SOI Effect (linear)

- SIS: $\hat{\beta} = -\mathbf{0.004}^{***}$ (se=0.001)
- Theory consistent: SOI > 0 (La Niña) $\rightarrow$ lower temp. (hiatus?!)

Breaks

- Could correct SST record for '**bucket bias**'

Non-linearities

- Non-linear transformations of trend retained

Automatic selection well developed for linear models

Algorithm can be extended to handle non-linear functions – not yet automated

- Test of functional form against linear
- Low order approximations for a range of bases
- Polynomials and exponentials $\times$ polynomials very general
- Indicator saturation to remove extreme observations/level shifts
- Number of potential non-linear variables large: choice of significance level
- Methods for more variables than observations
- Encompassing tests against specific functional forms

Complexity of empirical modeling – but theory consistent, empirically congruent model can be developed.

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Joint modelling of dynamics, location shifts, relevant variables and non-linearities essential.
Automatic model selection despite $N > T$ seems a viable approach to tackling all complications jointly.

- IIS and SIS do not preclude finding non-linearites, and non-linearities removed indicators found in linear specifications.
- Not removing the large outliers could hide the presence of other variables, including the non-linearities.
- Indicator saturation: insurance mechanism in non-linear modelling.

Castle, J. L., J. A. Doornik, D. F. Hendry, and F. Pretis (2015).
Detecting location shifts during model selection by step-indicator saturation.
Econometrics 3(2), 240–264.

Castle, J. L. and D. F. Hendry (2010).
A low-dimension portmanteau test for non-linearity.
Journal of Econometrics 158(2), 231–245.